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WEAPONS ASSIGNMENTS IN A TIERED AIMPOINT SYSTEM WHICH SATISFY D--ETC(U)
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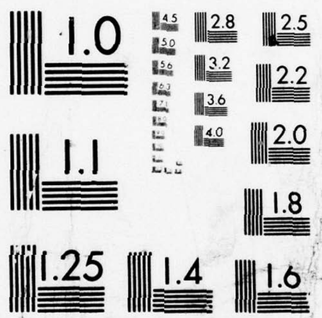
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(6) WEAPONS ASSIGNMENTS IN A TIERED AIMPOINT SYSTEM WHICH SATISFY DISPERSION CONSTRAINTS.

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TABLE OF CONTENTS

RESEARCH OBJECTIVES

When one is unable to place the available weapons to cover every installation so as to attain a minimum specified damage on each installation, the supply of weapons may be considered as a scarce resource which must be assigned to targets in such a way as to optimize a prescribed objective. The prescribed objective may be strictly military or economic or a composite of various intentions. On the other hand, when it is possible to sufficiently fulfill the objective, it may be advantageous to obtain the assignment of weapons which is optimal with respect to other criteria such as cost, reserves, and combat efficiency. Within the scope of this study, we consider a mixture of weapons and delivery vehicle types.

There are numerous factors and constraints which pertain to this assignment problem in its actual application, and such actual assignments require quite sizable investments of manpower and computation time. This research addresses the mathematical modeling analysis and solution of the assignment problem focusing on damage evaluation and supply, range, and coverage constraints as well as the dispersion restrictions on the weapon carried in any single delivery vehicle. The solution of the model is not intended as a blueprint for actual assignments by strategic units, but rather by concentrating just on these factors we intend the model as an analytical tool to assess the present detailed assignment algorithms and to evaluate future policy concerning nuclear weapon and delivery system procurement, deployment, targeting, and strategic arms limitation.

Accordingly, this research is directed toward a model which affords a reasonably quick solution algorithm; thus, the algorithm may be used to give broad consideration to many hypothetical alternatives, to provide a starting point from which a detailed blueprint for actual assignments by strategic units is constructed, or to generally evaluate present assignment schemes.

STATUS OF RESEARCH

At the beginning of this research the problem was presented in raw form. Although many important components had been identified, these components had not been characterized mathematically and there was no comprehensive framework for synthesis and analysis. Thus, the initial effort was concentrated on defining and mathematically characterizing the relevant features of the tiered aimpoint system, weapons, delivery vehicles, and a general value system. These characterizations led to a preliminary system of constraints.

1. Each delivery vehicle carried a fixed number of weapons.
2. The weapons could only be assigned to DGZs of matching type within the tiered aimpoint system.
3. Only DGZs in range of the delivery vehicle may be assigned weapons.
4. The weapons carried by any single delivery vehicle can only be disbursed within the footprint of that vehicle.

The consideration of range resulted in the Mercator computation of accessibility of weapon to DGZ. While the footprint of any delivery vehicle can be described as a complicated amalgam of physical processes, such computations have proved to be the major obstacle in providing the desired quick, analytical tool. Herein we develop the parallelogram footprint approximation of the true footprint; this device allows the consideration of footprint in the model without excessive computational burdens. While not an exact description of reality, the parallelogram footprint approximation is reasonable for the purposes of the quick, analytical tool.

Accordingly, the task became defined as finding the assignments which obtain the optimal or near optimal level of value subject to constraints. The notion of value extracted due to the assignment of any particular weapon, however, was obscured by the possible interactive effects of various weapons. This confounding was minimized by the addition of dummy DGZs to resolve the cases in which many weapons are assigned to one of the original DGZs or in which some of the assigned weapons cover the same installations. While this adds constraints to the problem, it eliminates the nonadditivity of the values extracted by each weapon. Dummy DGZs allow some interactive effects among weapons, yet, albeit subtly, allow the separate extracted values to be added to provide a measure of utility of an entire assignment.

The result of this modeling and approximation admitted a mixed integer/linear programming mathematical synopsis of the problem. While the simpler problem without dispersion restrictions can be solved using branch-and-bound with linear programming as the fathoming device since that problem structure allows quick access to a feasible integral optimun, an examination of some small sample problems revealed that this algorithm would be slow in the cases including dispersion restrictions. Problems of this type contain a large number of variables. Identification of decompositions of the problem by delivery vehicles enabled the construction of primal and dual solution algorithms which converge to the optimal assignment. So even if the solution procedure is truncated we obtain a feasible assignment and a measure of its closeness to optimality. In view of the analytical objectives, this is a reasonable result.

LIST OF WRITTEN PUBLICATIONS IN TECHNICAL JOURNALS

"A model of weapons assignment including dispersion constraints" to be submitted to Operations Research or Naval Research Logistics Quarterly is in preparation by Jerren Gould.

PROFESSIONAL PERSONNEL ASSOCIATED WITH THE RESEARCH EFFORT

Principal Investigator:

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INTERACTIONS

1. "Optimal weapon assignment in a tiered aimpoint system," (with Capt. S. J. Monaco) working group paper, 40th Military Operations Research Society Symposium, 14 December 1977, Monterey, California.

2. Consultative visit to FJSRL/NH included presentation of a parallelogram approximation of footprint to Lt. C. John Brush, Capt. S.J. Monaco, and Maj. Warren Langley. Briefings were provided on alternative value systems and new methods of generating the tiered aimpoint system. 10-11 April 1978.

3. Consultative visit HAF/SA and HAF/SASM included presentation of decomposition by delivery vehicle method and some computational results on small problems to Capt. Bert Knight, Capt. S.J. Monaco, and some of staff at HAF/SA. 28-29 June 1978.

INTRODUCTION

There are various types of delivery vehicles in which nuclear weapons are conveyed to a target. Any individual delivery vehicle may be categorized according to:

1. launch point
2. range
3. number of weapons conveyed
4. yield of each weapon
5. restrictions on the dispersion of the weapons

Based on these characteristics, it may be determined whether a particular collection of aimpoints are accessible to the weapons in a given delivery vehicle. All of the weapons carried in a single delivery vehicle must be targeted in a neighborhood of the others. For MIRVed ballistic missiles such a neighborhood is called a footprint; for manned bombers a reasonable flight path must be demonstrable. For delivery vehicles which carry a single weapon, the criterion reduces trivially to range.

The domain of a potential adversary contains numerous installations (airfields, missile silos, factories, etc.) and each installation has been appointed a value according to a given system of utility. There are several levels of weapon yield, called tiers. The value extracted from an installation depends on the tier of the weapon assigned to cover that installation as well as the location of the aimpoint relative to the location of the installation.

The main treatment herein is the case in which there are insufficient numbers of weapons allocated to a strike to extract value from every installation. Such a scenario is called a target-rich environment. The analysis and solution of other important scenarios follow analagously.

TIERED AIMPOINT SYSTEM

The supply of available weapons consists of weapons of various yields. The tier of a weapon corresponds to its yield inversely. That is, the lowest tier corresponds to the highest yield and the highest tier corresponds to the lowest tier; we shall assume that the tier ranges over the integers 1 to T. The tiered aimpoint system is a device to cope with the diversity of weapon yields. For each tier a collection of potential aimpoints, or designated ground zeros (DGZ), is generated.

The domain of a potential adversary is divided into complexes (target islands). A complex is a region containing installations such that a hit by a weapon at any point within the region will not affect installations in any other complex. The location of tier 1 DGZs are determined so as to partition the collection of installations in the complex into subcomplexes. The number and layout of these DGZs are constructed so as to obtain at least a threshold fraction of value extracted subject to prudence and parsimony. Within each subcomplex defined by the tier 1 DGZs, the higher tier DGZs are determined, tier by tier, in the same manner as the tier 1 DGZs are found within the complex. The value extracted by the weapons of any higher tier in a subcomplex must approximate the value extracted by the tier 1 weapon. Various parameters, such as hardness of any installation, affect the layout of the DGZs in any tier and heights of burst. An adaptation of Cooper's algorithm [13] may be used to define the layout of DGZs.

Certain hardened installations may require more than one weapon of the same tier to extract sufficient value. In such a case we consider this single installation as several installations. Despite the fact that these dummy installations share a common location, they may be partitioned into distinct DGZs; thus, each such dummy installation may be covered by a weapon without the others necessarily being covered. The values extracted from these dummy installations are constructed so as to incorporate the cooperative effects of multiple coverage. For example, the values extracted from the additional dummy installations may represent the marginal value extracted due to the assignment of the additional weapons which cover the real installation.

We label all of the DGZs in the layout as $j = 1,\ldots,N$. Each DGZ has a unique tier even though the coordinates of several aimpoints are identical. Let $t_2(j)$ represent the tier of DGZ j. Let $I(j)$ be the set of installations covered by DGZ j. By construction, for each tier t, $\{I(j) \mid j \in t_2^{-1}(t)\}$ is a partition of the set of installations, I. Also, if $t_2(j) = 1$ and $j \in I(j')$, then $I(j') \subset I(j)$; that is, the higher tier DGZs are nested within the subcomplex.

The layout of DGZs is constructed without regard to the delivery system deployment, dispersion characteristics, and inventory of weapons. If there is no assignment of weapons to DGZs so that sufficient value is extracted from every installation, we have a target-rich environment. Otherwise, we have a weapon-rich environment.

DELIVERY VEHICLES

We shall assume that the weapons conveyed within each delivery vehicle are identical. We label the delivery vehicles $k = 1,\ldots,K$. Let μ_k be the number of weapons, or reentry vehicles (RV), carried by delivery vehicle k, and let $t_1(k)$ represent the common tier of weapons carried by delivery vehicle k.

Each DGZ is constructed to accept a weapon of only one particular tier. That is, a weapon from delivery vehicle k may be assigned to DGZ j only if $t_1(k) = t_2(j)$. The ranges of all the weapons in delivery vehicle k are identical. Let R_k be the common range of weapon in delivery vehicle k. Since the locations of the launch site of delivery vehicle k and the DGZ j are known, by spherical trigonometry and consideration of the rotation of the earth during flight we can calculate the distance d_{kj} between the launch site k and DGZ j. It is a necessary condition for a weapon to be assigned from delivery vehicle k to DGZ j that both the range criterion be satisfied and the tiers of the weapon and DGZ match. We define the accessibility index

$$\alpha_{kj} = \begin{cases} 1 & \text{if } t_1(k) = t_2(j) \text{ and } d_{kj} \le R_k \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, d_{kj} need only be calculated for delivery vehicles and DGZs with matching tiers. Another computational saving may be accrued whenever similar delivery vehicles carrying a common tier of weapon are located in a relatively small neighborhood of each other as compared to the flight distances; in that case we can say that the delivery vehicles share a

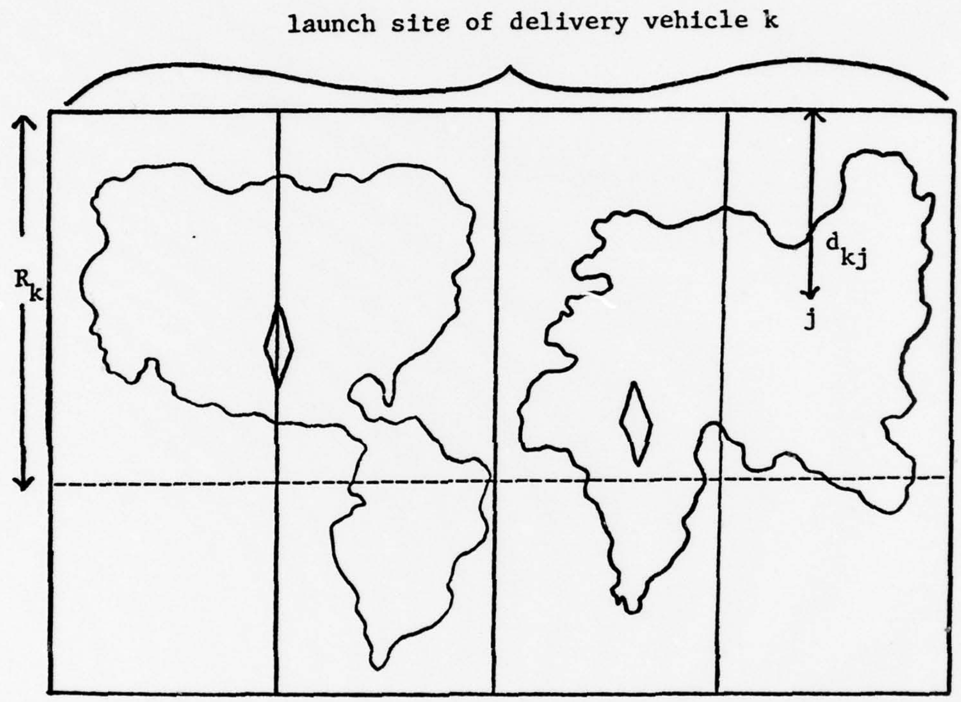

Figure 1 Mercator Projection with the North Pole as the Launch Site

common launch site. We can analogously find a representative launch site for the SLBMs by considering the centroid of any patrol region.

If one wishes to ignore time of flight, the computation of d_{kj} may be simplified. For the launch site of each delivery vehicle, or for each launch field, the map of the world is transformed by rectifying the great circles through the launch site in a manner akin to a Mercator projection. In such a case, d_{kj} is readily obtained as the latitude coordinate of DGZ j in the new map with the launch site as a pole.

VALUE

If a weapon is assigned to DGZ j, the tier of the weapon must match the tier of the DGZ. By construction, v_{ji}, the value extracted from installation i when a weapon is assigned to DGZ j, is known. These values extracted depend on the value of the installation, its hardness, and its geographic relationship to the aimpoint. Random variations due to aiming and reliability uncertainties are incorporated by fixing v_{ji} as the expected value extracted. It is possible that damage may be incurred to installations not covered by the DGZs, but we shall assume that, in terms of value extracted, this damage is negligible; that is, $v_{ji} = 0$ whenever $i \notin I(j)$. Due to certain strategic/political considerations, certain installations must not be damaged. For each DGZ j which covers such a "flagged" installation, we fix $v_{ji} \ll 0$ to insure that any reasonable weapons assignment would deny weapons to any DGZ convering installation i.

Let

$$a_{kj} = \begin{cases} 1 & \text{if a weapon from delivery vehicle } k \text{ is assigned to DGZ } j \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the decision variables a_{kj} are the unknowns we must determine for a solution of the problem.

The values extracted from distinct installations are assumed to be additive. However, if weapons are assigned to DGZs which cover the same installation (because of dummy DGZs this will only happen if the weapons are of distinct tiers), the total value extracted from the installation should not be the sum of the separate extracted values. Let $J(i) = \{j | i \in I(j)\}$ be the set of DGZs which cover the installation i. A reasonable measure of the value extracted from installation i is the greatest value extracted from that installation for any weapon assigned to a DGZ which covers installation i; that is,

$$\max_{k,j \in J(i)} a_{kj} v_{ji}.$$

Accordingly, in a target-rich environment we wish to find the feasible assignment $\{a_{kj}\}$ which maximizes

$$\sum_{i} \max_{k,j \in J(i)} a_{kj} v_{ji}. \tag{1}$$

Thus, only assignments which conduce a largest value extracted for some installation contribute to the objective; hence, others should be reassigned elsewhere. Notice that the a priori creation of dummy installations for certain targets allows some value extracted to accrue when the target is covered more than once.

Thus, in any optimal assignment every DGZ which receives a weapon must contain an installation from which it extracts the dominant value. When an installation is hit more than once, according to the objective (1) some weapons effectively extract no value from the installation. Although it is possible for this case to occur in an optimal solution, in a target-rich environment this waste should rarely occur. That is, almost all assignments will be to DGZs in which there is no competition among the weapons to extract value from the installations covered. Hence, as an approximation we exclude such competition by forbidding assignment schemes in which any installation is covered more than once. This constraint is

$$\sum_{k,j \in J(i)} a_{kj} \leq 1 \quad \forall i \tag{2}$$

Accordingly, the objective (1) transforms to

$$\sum_{i} \max_{k,j \in J(i)} a_{kj} v_{ji} = \sum_{k,j} a_{kj} v_j, \tag{3}$$

where $v_j = \sum_{i \in I(j)} v_{ji}$ is the value extracted from all of the installations in DZG j when an appropriately tiered weapon is so assigned. In a target-rich environment we may presume that all of the weapons allocated to a strike will be used.

DISPERSION

The dispersion problem arises because the weapons from a single delivery vehicle cannot be assigned arbitrarily, but rather each must be assigned to a DGZ "near" to the other DGZs receiving a weapon from the same delivery vehicle.

Each weapon carried in a MIRVed ballistic missile is first conveyed by the delivery vehicle until it is near the target region; each weapon separates from the delivery vehicle and is finally conveyed to the assigned DGZ (aimpoint) by a reentry vehicle. The region into which these weapons must be aimed is called the footprint of the delivery vehicle.

While the true footprint is somewhat amorphous because it depends on many factors which cannot and should not be considered in this analysis, it is generally somewhat elliptical. We shall assume that the boundary of the footprint is truly an ellipse with an axis parallel to a great circle through the launch point. The approximation of the elliptical footprint by a parallelogram footprint allows a mathematically more convenient characterization of the dispersion constraints for MIRVs. The parallelogram footprint allows a linear form of the dispersion constraints, whereas other characterizations of the footprint constraints are either inherently nonlinear or introduce more variables into the analysis.

Parallelogram footprint

Suppose delivery vehicle k is directed to (X_k,Y_k), the midpoint of its footprint. The coordinates (X_k,Y_k) are variables to be determined in the optimization. Let ρ_k be half of the downrange reach and ρ_k/θ_k be half of the cross range reach of the parallelogram footrpint characterization of delivery vehicle k. Let (x_j,y_j) be the coordinates of DGZ j. DGZ j will be within the parallelogram footprint if and only if (X_k,Y_k) is chosen so that

$$\theta_k|x_j-X_k| + |y_j-Y_k| \le \rho_k. \tag{4}$$

Let us define

$$e_{jk} = \begin{cases} x_j-X_k & \text{if } x_j \ge X_k \\ 0 & \text{if } x_j < X_k \end{cases}$$

and (5)

$$f_{jk} = \begin{cases} 0 & \text{if } x_j \ge X_k \\ X_k-x_j & \text{if } x_j < X_k \end{cases}$$

Clearly, e_{jk} and f_{jk} are nonnegative and not simultaneously positive. Clearly, $e_{jk}+f_{jk} = |x_j-X_k|$. Similarly, we define

$$g_{jk} = \begin{cases} y_j-Y_k & \text{if } y_j \ge Y_k \\ 0 & \text{if } y_j < Y_k \end{cases}$$

and (6)

$$h_{jk} = \begin{cases} 0 & \text{if } y_j \ge Y_k \\ Y_k-y_j & \text{if } y_j < Y_k \end{cases}$$

and observe also that g_{jk} and h_{jk} are nonnegative, not simultaneously positive, and $g_{jk}+h_{jk} = |y_j-Y_k|$.

Consider the following constraints to represent the parallelogram footprint constraints for the delivery vehicles

$$X_k + e_{jk} - f_{jk} = x_j \qquad \forall j,k \tag{7}$$

$$Y_k + g_{jk} - h_{jk} = y_k \qquad \forall j,k \tag{8}$$

$$\theta_k(e_{jk}+f_{jk}) + (g_{jk}+h_{jk}) - \rho_k a_{kj} - U_k(1-a_j) \le 0 \quad \forall j,k \tag{9}$$

where U_k is a suitably large number. It can be shown that when these constraints are embedded within the total model that there are algorithms which can guarantee nonnegative solutions for e_{jk}, f_{jk}, g_{jk}, h_{jk} with the appropriate pairs not simultaneously positive. Hence, the definitions (5,6) and (7,8) of e_{jk}, f_{jk}, g_{jk}, h_{jk} coincide. Since a_{kj} shall be constrained to 0 or 1, these constraints (7-9) admit two cases:

(1) $a_{kj} = 1$; here constraints (7-9) are equivalent to the fact that DGZ j must lie within the paralellogram footprint of delivery vehicle k as it must if such an assignment is to be made.

(2) $a_{kj} = 0$; since a weapon from delivery vehicle k is not assigned to DGZ j, it is irrelevant whether DGZ j lies in the parallelogram footprint. Here constraints (7-9) merely serve to reflect the fact that DGZ j lies within a much larger parallelogram. So if U_k is chosen sufficiently large, these constraints will always be satisfied when $a_{kj} = 0$.

Finding the best parallelogram footprint approximation to the true footprint

For our analysis here we shall consider the midpoint 0 as the origin of a Cartesian coordinate system. There are two types of errors that may occur in the designation of any footprint for analytic purposes. Let P denote the set of points within the parallelogram and T denote the set of points within the true footprint. We have a type 1 error for the points in $T-P$; roughly speaking, a type 1 error implies we have possibly prohibited making an assignment because of the mathematical formulation when, in fact, the assignment is physically realizable. On the other hand, we have a type 2 error for the points in $P-T$; a type 2 error implies that we have possibly permitted an assignment which is not physically realizable.

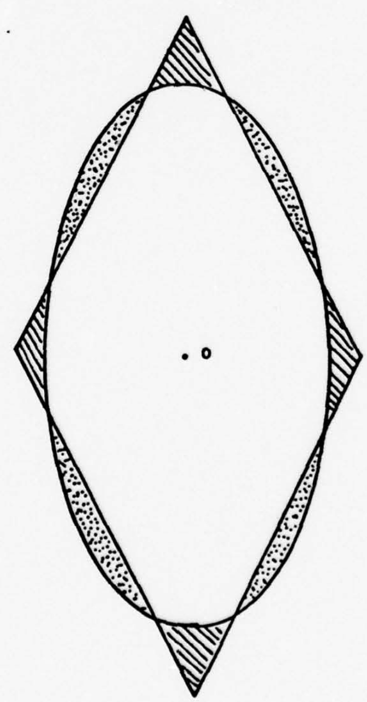

Figure 2 Comparison of true footprint and parallelogram footprint with coincident midpoints

Let c_1 be the loss associated with a unit area of type 1 error and c_2 be the loss associated with a unit area of type 2 error. Hence, a measure of closeness of the parallelogram approximation is

$$C(P) = c_1 \times \text{area of } (T-P) + c_2 \times \text{area of } (P-T). \tag{10}$$

Thus, one can find a best parallelogram footprint approximation by solving for $\hat{P}$ such that $C(\hat{P}) \le C(P)$ for all parallelograms P with midpoint 0.

Let the true footprint be an ellipse with half downrange reach r and half crossrange reach r/σ. Thus, the boundary of the true (elliptical) footprint is $\sigma^2x^2+y^2 = r^2$. The boundary of a parallelogram footprint with half downrange reach ρ and half crossrange reach ρ/θ may be represented in the first quadrant as $y=-\theta x+\rho$. Let $q = (\sigma^2r^2+\theta^2r^2-\sigma^2\rho^2)^{1/2}$. The points of intersection in Figure 3 are

$$(x_1,y_1) = \frac{1}{\sigma^2+\theta^2} (\theta\rho+q, \sigma^2\rho-\theta q) \tag{11}$$

$$(x_2,y_2) = \frac{1}{\sigma^2+\theta^2} (\theta\rho-q, \sigma^2\rho+\theta q) \tag{12}$$

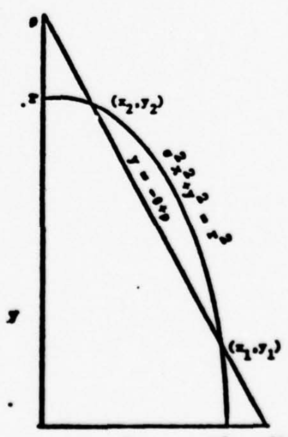

Figure 3 Quadrant of comparison of elliptic and parallelogram footprints

$$\text{Hence,}\quad C(\rho,\theta) = 2c_1\{\frac{r^2}{\sigma}\,[\tan^{-1}(\frac{y_2}{\sigma x_2}) - \tan^{-1}(\frac{y_1}{\sigma x_1})] + \frac{y_1\rho}{\theta} + x_2\rho - \frac{\rho^2}{\theta}\}$$

$$+ 2c_2\{x_2\rho + \frac{y_1\rho}{\theta} - \frac{r^2}{\sigma}\,[\tan^{-1}(\frac{y_1}{\sigma x_1}) + \tan^{-1}(\frac{\sigma x_2}{y_2})]\} \qquad (13)$$

To find the best parallelogram $P(\hat{\rho},\hat{\theta})$ we must solve the system of equations

$$\frac{\partial C(\hat{\rho},\hat{\theta})}{\partial\hat{\rho}} = 0 \qquad \frac{\partial C(\hat{\rho},\hat{\theta})}{\partial\hat{\theta}} = 0. \qquad (14)$$

Such a solution may only be attempted numerically given the values of c_1, c_2, r, and σ. Accordingly, we introduce a simpler alternative method with a different measure of closeness.

Here we restrict the problem by fixing $\sigma=\theta$; that is, the ratio of major axis to minor axis of the ellipse and the parallelogram are identical. We wish to find the value of ρ which minimizes the maximal distance disparity between the boundaries of the elliptical and parallelogram footprints. By disparity we mean the shortest distance as measured from any fixed point on one footprint boundary to the other footprint boundary. By symmetry we need only consider the first quadrant. From Figure 4 it becomes clear after some analysis that the maximum is obtained as either the length of AB, CD, or EF. From analytic geometry we find

length AB	$\dfrac{\sqrt{2}\,r-\rho}{(1+\sigma^2)^{1/2}}$
length CD	$\rho - r$
length EF	$(\rho-r)/\sigma$.

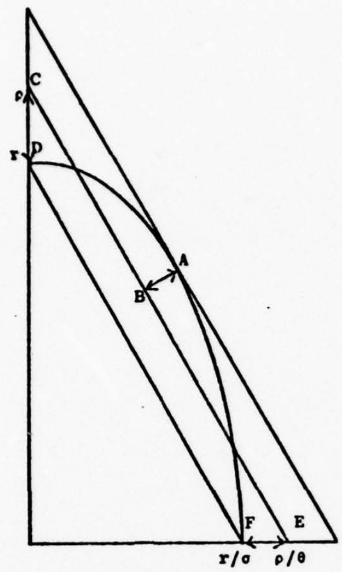

Figure 4 Minimax criterion for determining the best parallelogram approximation (σ=θ)

Hence, the solution for ρ which minimizes the maximum distance disparity reduces to the following linear programming problem:

$$\text{minimize } E \tag{15}$$

$$\text{subject to} \quad E \geq \frac{\sqrt{2}\, r-\rho}{(1+\sigma^2)^{1/2}} \tag{16}$$

$$E \geq \rho-r \tag{17}$$

$$E \geq (\rho-r)/\sigma \tag{18}$$

$$r \leq \rho \leq \sqrt{2}\, r. \tag{19}$$

The value of ρ which achieves the optimum is

$$\tilde{\rho} = r \, \frac{[(1+\sigma^2)^{1/2} + \sqrt{2}\min(1,\sigma)]}{[(1+\sigma^2)^{1/2} + \min(1,\sigma)]} \, . \tag{20}$$

So according to this criterion the best parallelogram footprint approximation to the true footprint $\sigma^2x^2+y^2 = r^2$ is given by half downrange reach $\tilde{\rho}$ and half crossrange reach $\tilde{\rho}/\sigma$. In applications r and σ depend on the properties of delivery vehicle k; hence, $\tilde{\rho}$ may depend on k. In considering assignments the footprints are translated so that the midpoint of the footprint of delivery vehicle k is (X_k, Y_k).

MATHEMATICAL STATEMENT OF THE PROBLEM FOR BALLISTIC MISSILES

We wish to find assignments of weapons to DGZs which do well (optimize) the total value extracted subject to the constraints of accessibility, avoidance of multiple coverage (with the a priori exceptions due to dummy DGZs), and dispersion due to footprint. Let μ_k denote the number of weapons carried by delivery vehicle k; clearly, we must also include the constraint (23) that no more than μ_k weapons from delivery vehicle k be assigned to DGZs.

Formally, then, we obtain the mixed integer/linear programming problem

$$\text{maximize} \sum_{j=1}^{N} \sum_{k=1}^{K} a_{kj} v_j \tag{21}$$

$$\text{subject to} \quad a_{kj} \le \alpha_{kj} \qquad \forall j,k \tag{22}$$

$$\sum_{j=1}^{N} a_{kj} \le \mu_k \qquad \forall k \tag{23}$$

$$\sum_{k=1}^{K} \sum_{j \in J(i)} a_{kj} \le 1 \qquad \forall i \tag{24}$$

$$X_k + e_{jk} - f_{jk} = x_j \qquad \forall j,k \tag{25}$$

$$Y_k + g_{jk} - h_{jk} = y_j \qquad \forall j,k \qquad (26)$$

$$\theta_k(e_{jk} + f_{jk}) + (g_{jk} + h_{jk})$$
$$-\rho_k a_{kj} - U_k(1-a_{kj}) \leq 0 \qquad \forall j,k \qquad (27)$$

$$a_{kj}, e_{jk}, f_{jk}, g_{jk}, h_{jk} \qquad \forall j,k \qquad (28)$$

$$a_{kj} \text{ integer} \qquad \forall j,k \qquad (29)$$

Since a_{kj} is constrained to be a nonnegative integer, constraint (24) further implies that a_{kj} is either 0 or 1, as is desired. It becomes evident now that there exists a feasible assignment, namely $a_{kj} = 0\ \forall k,j$; this assignment, however, accrues no value. The objective function (21) drives the problem to find the maximal total value extracted subject to the constraints. Since the set of feasible assignments is nonempty and finite, problem (21-29) has a solution.

Due to the large number of variables in problem (21-29) and the binary nature of each a_{kj}, the computational aspects of finding the solution are quite cumbersome. In view of the uses of the results, we may be satisfied with good suboptimal feasible assignment schemes. This approach is especially important in a target-rich environment in which a computer algorithm may spend the majority of time finding an optimal assignment with small marginal return over a good suboptimal solution reached quickly.

A DUAL SOLUTION METHOD

Herein we shall incorporate the accessibility constraints (22) by removing all variables subscripted $_{jk}$ or $_{kj}$ whenever $\alpha_{kj} = 0$. Let $r_k = \sum_j \alpha_{kj}$ be the number of DGZs accessible to the weapons of delivery vehicle k. For each k we may define the functions $k(\ell) = (k, j_\ell)$, where j_ℓ is the ℓ^{th} DGZ accessible to delivery vehicle k according to some ordering. We now introduce the following notation:

$$\underline{a}_k' = (a_{k(1)}, \ldots, a_{k(r_k)})$$

$$\underline{v}_k' = (v_{k(1)}, \ldots, v_{k(r_k)})$$

$$\underline{x}_k' = (x_{k(1)}, \ldots, x_{k(r_k)})$$

$$\underline{y}_k' = (y_{k(1)}, \ldots, y_{k(r_k)})$$

$$\underline{z}_k = (X_k, Y_k, e_{k(1)}, f_{k(1)}, g_{k(1)}, h_{k(1)}, \ldots, g_{k(r_k)}, h_{k(r_k)}).$$

Note that the components of $\underline{v}_k$, $\underline{x}_k$, and $\underline{y}_k$ do not depend on the first component of $k(\ell)$. Suppose that m of the constraints (24) are non-redundant. Let Q_k be the $m \times r_k$ matrix with element (c,d) being 1 if DGZ k(d) appears in the c^{th} nonredundant constraint of (24). Accordingly, the mathematical statement may be rewritten as:

$$\max \sum_{k=1}^{k} \underline{v}_k' \underline{a}_k \tag{30}$$

$$\text{subject to} \sum_{k=1}^{k} Q_k \underline{a}_k \le \underline{b}_o = \underline{J}_m \tag{31}$$

$$\begin{pmatrix} \underline{J}'_{r_k} & \vdots & \underline{0} \\ (U_k - \rho_k)\ I_{r_k} & \vdots & D_k \\ 0 & \vdots & E_k \end{pmatrix} \begin{pmatrix} \underline{a}_k \\ \cdots \\ \underline{z}_k \end{pmatrix} \begin{matrix} \le \\ \le \\ = \\ = \end{matrix} \begin{pmatrix} \mu_k \\ U_k \underline{J}_k \\ \underline{x}_k \\ \underline{y}_k \end{pmatrix} = \underline{b}_k \qquad \forall_k \tag{32}$$

$$\underline{z}_k \ge \underline{0} \qquad \forall k \tag{33}$$

$$\underline{a}_k \text{ binary } \forall_k. \tag{34}$$

$$\text{Let } F = \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_k \\ \underline{J}_{r_1} & 0 & & 0 \\ (U_1 - \rho) I_{r_1} & \cdot & & \cdot \\ 0 & 0 & & \\ & \underline{J}_{r_2} & & \\ & (U_2 - \rho) I_{r_2} & & \\ & 0 & \ddots & \\ & \cdot & & 0 \\ & \cdot & & \underline{J}_{r_k} \\ & \cdot & & (U_k - \rho_k) I_{r_k} \\ 0 & 0 & & \end{pmatrix}$$

$$\text{and } H = \begin{pmatrix} 0 & 0 & & \\ \underline{o} & \underline{o} & & \\ D_1 & 0 & & 0 \\ E_1 & 0 & & \\ & D_2 & & \\ & E_2 & \ddots & \\ 0 & & & \\ & & & D_k \\ & & & E_k \end{pmatrix}.$$

For $\underline{A} = \begin{pmatrix} \underline{a}_1 \\ \vdots \\ \underline{a}_k \end{pmatrix}$ and $\underline{Z} = \begin{pmatrix} \underline{z}_1 \\ \vdots \\ \underline{z}_k \end{pmatrix}$, the constraints (31, 32, 34)

may be represented as $F\underline{A} + H\underline{Z} \leqq \underline{b}$, where $\underline{b} = \begin{pmatrix} \underline{b}_o \\ \underline{b}_1 \\ \vdots \\ \underline{b}_k \end{pmatrix}$ and $\underline{A}$ is

binary. Let $F = \{ \underline{A} | \exists \underline{Z} \geq 0 \ni H\underline{Z} \leqq \underline{b} - F\underline{A}, \underline{A} \text{ binary}\}$. Hence, $\underline{A}$ is a feasible binary solution iff $\underline{A} \in F$.

Let $\underline{\eta} = \begin{pmatrix} \underline{\eta}_o \\ \underline{\eta}_1 \\ \vdots \\ \underline{\eta}_k \end{pmatrix}$, where $\underline{\eta}_o$ is $m \times 1$ and $\underline{\eta}_k = \begin{pmatrix} \alpha_k \\ \underline{\beta}_k \\ \underline{\gamma}_k \end{pmatrix}$ with

α_k, $\underline{\beta}_k$, $\underline{\gamma}_k$, 1×1, $r_k \times 1$, and $2r_k \times 1$, respectively, for $k = 1,\ldots,K$.

By Farkas' Lemma, $\exists \underline{Z} \geq \underline{0} \ni H\underline{Z} \leqq \underline{b} - F\underline{A}$ iff $(\underline{b} - F\underline{A})' \eta \geq 0$ whenever $H'\underline{\eta} \geq 0$, $\underline{\eta}_o \geq 0$, and $\alpha_k \geq 0$, $\underline{\beta}_k \geq \underline{0}$ for $k = 1,\ldots, K$. Let C be the cone of such values of $\underline{\eta}$.

By duality [1], we may reformulate the problem as

$$\max_{\underline{A} \in F} \left\{ \sum_{k=1}^{K} \underline{v}_k' \, \underline{a}_k + \min_{\underline{\eta} \in C} \{ (\underline{b} - F\underline{A})' \, \underline{\eta} \} \right\}. \tag{35}$$

C is nonempty. Moreover since a feasible solution to the original problem exists, the minimum in (35) is obtained at an extreme point of C.

Let C have extreme points $\underline{\eta}_i^{(e)}$, $i = 1,\ldots,n_e$, and extreme rays $\underline{\eta}_i^{(r)}$, $i = 1,\ldots,n_r$. Hence, we rewrite (35) as

$$\text{maximize } V \tag{36}$$

subject to

$$V \le \sum_{k=1}^{k} \underline{a}_k' \underline{v}_k + (\underline{b} - F\underline{A})' \underline{\eta}_i^{(e)}, \; i = 1,\ldots,n_e \tag{37}$$

$$(\underline{b} - F\underline{A})' \underline{\eta}_i^{(r)} \ge 0, \quad i = 1,\ldots,n_r \tag{38}$$

$$\underline{a}_k \text{ binary.} \tag{39}$$

Consider now a relaxation of (36-39)

$$\text{maximize } V \tag{40}$$

subject to

$$V \le \sum_{k-1}^{k} \underline{a}_k' \underline{v}_k + (\underline{b} - F\underline{A})' \underline{\eta}_i^{(e)} \; \forall\, i \in I_2 \tag{41}$$

$$(\underline{b} - F\underline{A})' \underline{\eta}_i^{(r)} \ge 0 \quad \forall\, i \in I_2 \tag{42}$$

$$\underline{a}_k \text{ binary,} \tag{43}$$

where I_1 and I_2 are subsets of the integers $1,\ldots,n_e$ and $1,\ldots,n_r$, respectively. In the usual way of considering relaxations, we may obtain an iterative procedure for solution of the original problem:

1. Initialize I_1 and I_2 with a few (or no) elements each.
2. Obtain V_D^o, $\underline{A}_D^o$ as the finite optimal solution to (40-43).

3. Solve $W = \min (\underline{b} - F\underline{A}_D^o)' \underline{\eta}$ (44)

subject to $H'\underline{\eta} \geq 0$ (45)

$\alpha_k \geq 0, \underline{\beta}_k \geq \underline{0} \qquad \forall\ k$ (46)

$\underline{\eta}_o \geq 0$ (47)

4. (a) If $W \leq 0$, then $\underline{A}_D^o$ is the solution to the original problem and the optimal subjective value is V_D^o. Stop.

(b) If $0 < W < \infty$, then (44-47) has a finite optimal solution at $\underline{\eta}_*^{(e)}$ with

$$V_D^o > \sum_{B=1}^{k} \underline{a}_k' \underline{v}_k + (\underline{b} - F\underline{A}_D^o)' \underline{\eta}_*^{(e)} . \tag{48}$$

The vector $\underline{\eta}_*^{(e)}$ is an extreme point of C not yet considered in (41). Enlarge I, so that $\underline{\eta}_*^{(e)}$ is considered in (41). Return to Step 2.

(c) If $W = \infty$, then (44-47) has an extreme ray $\underline{\eta}^{(r)}$ and an extreme point $\underline{\eta}^{(e)}$ such that the objective becomes unbounded along the half-line $\underline{\eta} = \underline{\eta}^{(e)} + \lambda\underline{\eta}^{(r)}$, $\lambda \geq 0$. Accordingly, $(\underline{b} - F\underline{A}_D^o)' \underline{\eta}^{(r)} < 0$. Hence, enlarge I_2 to include $\underline{\eta}^{(r)}$. If, in addition, (48) is true, also enlarge I_1 to include $\underline{\eta}^{(e)}$. Return to Step 2.

The finite convergence of this algorithm follows directly from the finite number of constraints in (36-39).

We may simplify problem (44-47) by decomposing it into sub-problems according to delivery vehicle since

$$(\underline{b}-F\underline{A}_D^o)'\,\underline{\eta} = (\underline{J}_m - \sum_{k=1}^{K} Q_k\underline{a}_k^o)'\,\underline{\eta}_o$$

$$+ \sum_{k=1}^{K} (\mu_k - \underline{J}_{r_k}\underline{a}_k^o)'\alpha_k$$

$$+ \sum_{k=1}^{K} (U_k\underline{J}_{r_k} - (U_k-\rho_k)\underline{a}_k)'\underline{\beta}_k$$

$$+ \sum_{k=1}^{K} (\underline{x}_k' \vdots \underline{y}_k')\,\underline{\Upsilon}_k \tag{49}$$

is separably additive. That is, solving (44-47) is equivalent to solving

$$\min\ (U_k\underline{J}_{r_k} - (U_k-\rho_k)\underline{a}_k^o)'\,\underline{\beta}_k + (\underline{x}_k' \vdots y_k')\,\underline{\Upsilon}_k \tag{50}$$

subject to

$$D_k'\underline{\beta}_k + E_k'\,\underline{\Upsilon}_k \geq \underline{0} \tag{51}$$

$$\underline{\beta}_k \geq \underline{0}, \tag{52}$$

$$\min\ (\mu_k - \underline{J}_{r_k}'a_k^o)\alpha_k \tag{53}$$

subject to $\alpha_k \geq 0,$ (54)

for each $k=1,\ldots, K,$ and

$$\min\ (\underline{J}_m - \sum_{k=1}^{K} Q_k\underline{a}_k^o)'\,\underline{\eta}_o \tag{55}$$

subject to $\underline{\eta}_0 \geq \underline{0}$.

The quantity W is the sum of the optimal values of the objectives of (50, 53, 55).

Problem (40-43) is a linear program with binary variables. Since successive problems (40-43) in the iteration differ by the addition of one or two constraints, a cutting-plane method [8, 11, 7, 9] may be efficient.

At any iteration V_D^o provides an upper bound on the objective value of the original problem. As we proceed through the iterations, V_D^o is nonincreasing. We, however, only obtain a feasible solution to the original problem when the iteration is terminated.

A PRIMAL SOLUTION METHOD

Let us adopt (30-34) as the form of the problem, but add the redundant constraints $\underline{a}_k \leq \underline{J}_{r_k}$ and rewrite (32) as

$$B_k \begin{pmatrix} \underline{a}_k \\ \underline{z}_k \end{pmatrix} \begin{matrix} \leq \\ = \end{matrix} \begin{pmatrix} \underline{J}_{r_k} \\ \underline{b}_k \end{pmatrix} = \underline{c}_k .$$

Let $S_k = \left\{ \begin{pmatrix} \underline{a}_k \\ \underline{z}_k \end{pmatrix} \middle| \; B_k \begin{pmatrix} \underline{a}_k \\ \underline{z}_k \end{pmatrix} \begin{matrix} \leq \\ = \end{matrix} \underline{c}_k, \begin{pmatrix} \underline{a}_k \\ \underline{z}_k \end{pmatrix} \geq \underline{0} \right\}$ have extreme points $\underline{\xi}_{\ell k} = \begin{pmatrix} \underline{\xi}_{\ell k}(a) \\ \underline{\xi}_{\ell k}(z) \end{pmatrix}$.

S_k is a bounded convex set; an arbitrary element can be written as a convex combination of its extreme points $\sum_{\ell} \underline{\xi}_{\ell k} \gamma_{\ell k}$, where $\sum_{\ell} \gamma_{\ell k} = 1$ and $\gamma_{\ell k} \geq 0$. Hence, the problem without the integer constraints may be stated as

$$V_p = \text{maximize} \sum_k \sum_{\ell} \gamma_{\ell k} \, \varphi_{\ell k} \tag{57}$$

subject to

$$\sum_k \sum_\ell \gamma_{\ell k} \underline{P}_{\ell k} + I_{mxm}\, \underline{s} = \underline{b}_o \tag{58}$$

$$\sum_\ell \gamma_{\ell k} = 1 \qquad \forall k \tag{59}$$

$$\gamma_{\ell k} \geq 0 \qquad \forall \ell, k, \tag{60}$$

where $\varphi_{\ell k} = \underline{v}_k' \underline{\xi}_{\ell k}(a)$ and $\underline{P}_{\ell k} = Q_k \underline{\xi}_{\ell k}(a)$ and $\underline{s}$ represents slack variables. The initial basis consists of $\underline{s} = \underline{J}_m$ and $\gamma_{11} = \ldots \gamma_{1K} = 1$ corresponding to the extreme points $\underline{\xi}_{11}(a) = \underline{0}, \ldots, \underline{\xi}_{1K}(a) = \underline{0}$. With proper choice of global coordinate system, we find the remaining non-negative components of extreme point $\underline{\xi}_{1k}$ as $e_{jk} = x_j$ and $g_{jk} = y_j$. The form of the master program (57-60) is readily amenable to generalized lower bounding, a specialization of the simplex method which solves such problems while maintaining a mxm "working basis" [5].

For working basis $\mathcal{B}$, let $(\underline{\omega}_1' : \underline{\omega}_o') = \underline{\phi}_{\mathcal{B}}' \mathcal{B}^{-1}$, where $\underline{\omega}_1$ consists of the dual variables corresponding to the m constraints (58) and $\underline{\omega}_o$ consists of the dual variables corresponding to the K constraints (59). Let $\underline{\delta}_k$ denote the Kx1 vector with 1 in the k^{th} row and 0 otherwise. Hence, the "added value" for including the variable $\gamma_{\ell k}$ into the basis is

$$\bar{\varphi}_{\ell k} = \varphi_{\ell k} - (\underline{\omega}_1' : \underline{\omega}_o') \begin{pmatrix} \underline{P}_{\ell k} \\ \underline{\delta}_k \end{pmatrix}$$

$$= [\underline{v}_k' - \underline{\omega}_1' Q_k]\, \underline{\xi}_{\ell k}(a) - \omega_{ok}. \tag{61}$$

Since $\underline{\xi}_{\ell k}(a)$ is the decision variable component of an extreme point of S_k, we may find the largest "added value" for any fixed k by solving

$$W_k = \max\ [\underline{v}_k' - \underline{\omega}_1' Q_k]\underline{a}_k - \omega_{ok} \qquad (62)$$

aubject to

$$B_k \begin{pmatrix} \underline{a}_k \\ \underline{z}_k \end{pmatrix} \begin{matrix} \le \\ = \end{matrix} \underline{c}_k \qquad (63)$$

$$\underline{a}_k \ge \underline{0}, \quad \underline{z}_k \ge \underline{0} \ . \qquad (64)$$

The "added value" for including the slack variable s_j into the basis is $-\omega_{1j}$. If $\max(\max_k W_k, \max_j (-\omega_{1j})) \le 0$, then we have obtained the optimal solution to the problem without the integer constraints. Otherwise, we enter into the basis the variable corresponding to the largest "added value." Note that the solution of (62-64) must occur at an extreme point of S_k. When $\underline{a}_k$ is constrained to be binary, for each k it can be arranged that there is a unique $\gamma_{\ell k}=1$ in constraint (59).

For each k, the problems (62-64) are much smaller than the original and, hence, the solutions are computationally easier to obtain. Therefore, it is here we reintroduce the integer constraints, $\underline{a}_k$ binary. By solving (62-64) with the additional constraints $\underline{a}_k$ binary by the method of cutting planes, we adaptively shrink S_k and may find new "extreme points" due to the cuts. Note that it is not necessary in the master program to have an a priori evaluation of all the extreme points $\underline{\xi}_k$, but rather to keep track of the current basis and entering extreme point as defined by (62-64) with $\underline{a}_k$ binary.

In this primal algorithm we start with a basis which yields a 0 objective value. At each iteration, the objective function is nondecreasing. While adding the constraints $\underline{a}_k$ binary at each iteration destroys the guarantee of convergence to the optimum of the original problem, at each iteration we have a *feasible, binary solution.*

NEAR OPTIMAL SOLUTIONS

At any iteration in the dual method V_D^o provides an upper bound on the total optimal value extracted; however, only at optimality is the corresponding binary $\underline{A}_D^o$ feasible. At any iteration in the primal method V_P^o provides a lower bound on the total optimal value extracted, but the corresponding $\underline{A}_P^o$ is feasible and binary. As the dual approaches optimality there is likely to be a large number of iterations with marginal reduction of total value extracted due to the target-rich environment. Also, in addition the primal method being computationally slower because of the combinatorially greater binary considerations in (62-64) for each k, there is no guarantee of convergence of the primal method to optimality. Hence, if V_P^o and V_D^o are relatively close we may terminate computation and accept $\underline{A}_P^o$ as a good suboptimal solution with total value extracted V_P^o. Alternatively, it may be possible to deassign some weapons from $\underline{A}_D^o$ to obtain a feasible binary solution with total value extracted better than V_P^o. The degree of closeness of V_P^o and V_D^o are directly related to the rates of convergence and the computational capability and patience of the user.

ALTERNATIVE PARALLELOGRAM FOOTPRINTS

In the model (21-29) previously presented we have restricted each individual delivery vehicle to only one possible footprint. The physical mechanism whereby weapons are thrust from delivery vehicles, however, allows a variety of potential footprint shapes ranging from prolate to oblate. Let

$$\sigma = \frac{\text{half downrange reach}}{\text{half crossrange reach}}.$$

For any fixed σ, using the minimax criterion with $\theta = \sigma$ the best parallelogram footprint approximation may be represented as (θ, ρ_θ).

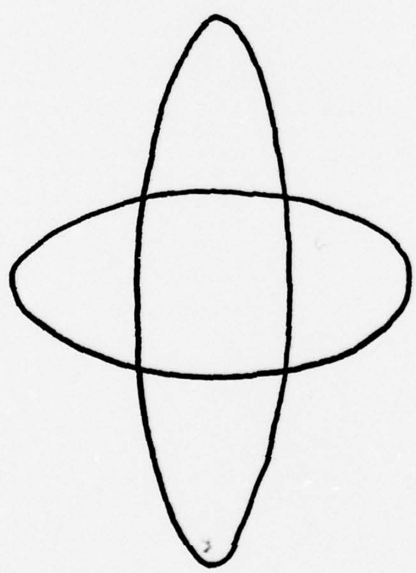

Figure 5 Oblate and Prolate Footprints

As we shall see, each possible value of θ for a delivery vehicle introduces a large number of additional variable. Furthermore, these footprints include much of the same area in common. Let ν_k be the number of potential footprints for delivery vehicle k. From the computational standpoint, the problem is already immense with $\nu_k = 1 \ \forall \ k$; however, if one uses $\nu_k=2$ for some k, the best values to consider are θ_1 (most oblate) and θ_2 (most prolate). One may also consider keeping $\nu_k=1 \ \forall \ k$, but give diversity to the single choices of θ_k for identical delivery vehicle types. Recall that it is only required that the relevant DGZs lie within the parallelogram footprint; their configuration within the footprint is irrelevant for the purposes of this investigation.

We represent the m^{th} potential parallelogram footprint as (θ_{km}, ρ_{km}). For $k=1,\ldots,K$ and $m=1,\ldots,\nu_k$, let q_{km} indicate whether delivery vehicle k uses its m^{th} potential parallelogram footprint. Also, let d_{kjm} indicate whether a weapon from delivery vehicle k is assigned to DGZ_j using footprint m. Note that a delivery vehicle may adopt only one of the potential footprints. Hence, we may generalize the model:

$$\text{maximize} \quad \sum_{k=1}^{K} \sum_{j=1}^{N} a_{kj} \, v_j \tag{65}$$

$$\text{subject to} \quad a_{kj} \le \alpha_{kj} \tag{66}$$

$$\sum_{m=1}^{\nu_k} d_{kjm} = a_{kj} \tag{67}$$

$$d_{kjm} \le q_{km} \tag{68}$$

$$\sum_{j=1}^{N} a_{kj} \leq \mu_k \tag{69}$$

$$\sum_{k=1}^{K} \sum_{j \in J(i)} a_{kj} \leq 1 \tag{70}$$

$$X_{km} + e_{jkm} - f_{jkm} = x_j \tag{71}$$

$$Y_{km} + g_{jkm} - h_{jkm} = y_j \tag{72}$$

$$\theta_{km}(e_{jkm} + f_{jkm}) + (g_{jkm} + h_{jkm}) \tag{73}$$

$$-\rho_{km} d_{kjm} - U(1-d_{kjm}) \leq 0 \tag{74}$$

$$\sum_{m=1}^{\nu_k} q_{km} \leq 1 \tag{75}$$

$$a_{kj},\ d_{kjm},\ e_{jkm},\ f_{jkm},\ g_{jkm},\ h_{jkm},\ q_{km} \geq 0 \tag{76}$$

$$a_{kj},\ q_{km},\ d_{kjm} \text{ integer.} \tag{77}$$

FLIGHT PATHS FOR MANNED BOMBERS

If delivery vehicle k is a manned bomber, then it proceeds from a given initial point, makes a tour of its assigned DGZs, and terminates at one of a set of designated airfields. If we allow a manned bomber a generous footprint and appropriately calculate a priori accessibilities we may defer evaluation of feasibility of the flight path to an external algorithm.

If the terminus is fixed then demonstration of flight path feasibility is mathematically equivalent to a traveling salesman problem. If the terminus is left variable, the airfield capacities must be observed. While such algorithms exist, common sense is likely to be quicker because μ_k is relatively small and there is a natural order to the tour. Furthermore, in a target-rich environment we may alter the assigned DGZs to make the flight path feasible or more sensible without much loss of total value extracted.

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NOTATION

$\underline{a}_k$	indicator of the DGZs assigned to weapons from delivery vehicle k
a_{kj}	indicator whether a weapon from delivery vehicle k is assigned to DGZ j
$\underline{A}$	vector of assignment variables for all delivery vehicles
$\underline{A}_D^o$	tentative dual assignment solution
$\underline{A}_P^o$	tentative primal assignment solution
C	cone of dual variables which yield primal feasible solutions
d_{kj}	distance between the launch point of delivery vehicle k and DGZ j
d_{kjm}	indicator of whether a weapon from delivery vehicle k is assigned to DGZ j using footprint m
e_{jk}	positive part of $x_j - X_k$
e_{jkm}	positive part of $x_j - X_{km}$
F	set of feasible binary assignments
f_{jk}	negative part of $x_j - X_k$
f_{jkm}	negative part of $x_j - X_{km}$
g_{jk}	positive part of $y_j - Y_k$
g_{jkm}	positive part of $y_j - Y_{km}$
h_{jk}	negative part of $y_j - Y_k$
h_{jkm}	negative part of $y_j - Y_{km}$
I	set of installations
I(j)	set of installations covered by DGZ j
I_n	n x n identity matrix
J(i)	set of DGZs which cover installation i
$\underline{J}_n$	n x 1 vector of unities

K	number of delivery vehicles
$k(\ell)$	a function returning k and the ℓ^{th} DGZ accessible to weapon from delivery vehicle k
N	number of DGZs
n_e	number of extreme points of C
n_r	number of extreme rays of C
P	region enclosed by parallelogram footprint
$\underline{P}_{\ell k}$	coverage induced by $\underline{\xi}_{\ell k}(a)$
Q_k	installation coverage indicator for DGZs accessible to weapons from delivery vehicle k
q_{km}	indicator of whether delivery vehicle k adopts potential parallelogram footprint m
r	half downrange reach of true footprint
r_k	number of DGZs accessible to weapons from delivery vehicle k
R_k	range of weapon from delivery vehicle k
S_k	polytope generated by constraints relative to delivery vehicle k
T	region enclosed by true footprint
t_1	a function such that $t_1(k)$ is the tier of weapon carried by delivery vehicle k
t_2	a function such that $t_2(j)$ is the tier of DGZ j
U_k	a suitably large number
v_j	value extracted from the installations in DGZ j
$\underline{v}_k$	values of the DGZs accessible to weapons from delivery vehicle k
v_{ji}	value extracted from installation i when a weapon is assigned to DGZ j

V_D^o	tentative dual objective value
V_P^o	tentative primal objective value
$\underline{x}_k$	crossrange coordinates of DGZs accessible to weapons from delivery vehicle k
(x_j, y_j)	coordinates of DGZ j
(X_k, Y_k)	midpoint of the parallelogram footprint of delivery vehicle k
(X_{km}, Y_{km})	midpoint of the $m^{\underline{th}}$ potential parallelogram footprint of delivery vehicle k
$\underline{y}_k$	downrange coordinates of DGZs accessible to weapons from delivery vehicle k
$\underline{Z}$	vector of dispersion variables for all delivery vehicles
$\underline{z}_k$	midpoint of the parallelogram footprint of delivery vehicle k and differences to coordinates of DGZs accessible to weapons from delivery vehicle k
α_k	dual variable corresponding to the capacity constraint for delivery vehicle k
α_{kj}	accessibility indicator for weapons from delivery vehicle k to DGZ j
$\underline{\beta}_k$	vector of dual variables corresponding to the parallelogram footprint constraints for delivery vehicle k
$\underline{\gamma}_k$	vector of dual variables corresponding to constraints of location of the footprint of delivery vehicle k
$\gamma_{\ell k}$	indicator of inclusion of $\underline{\xi}_{\ell k}$
$\underline{\delta}_k$	$k^{\underline{th}}$ column of I_K
$\underline{\eta}$	vector of dual variables
$\underline{\eta}_o$	vector of dual variables corresponding to the single coverage constraints
$\underline{\eta}_k$	vector of dual variables corresponding to capacity and dispersion constraints for delivery vehicle k

$\underline{\eta}_i^{(e)}$	an extreme point of C
$\underline{\eta}_i^{(r)}$	an extreme ray of C
θ_k	ratio of downrange reach to crossrange reach of the parallelogram footprint delivery vehicle k
θ_{km}	ratio of downrange reach to crossrange reach of m^{th} potential footprint of delivery vehicle k
μ_k	number of weapons in delivery vehicle k
ν_k	number of potential parallelogram footprints of delivery vehicle k
$\underline{\xi}_{\ell k}$	an extreme point of S_k
$\underline{\xi}_{\ell k}(a)$	elements of $\underline{\xi}_{\ell k}$ corresponding to the assignment variables of delivery vehicle k
$\underline{\xi}_{\ell k}(z)$	elements of $\underline{\xi}_{\ell k}$ corresponding to the location of the footprint of delivery vehicle k
ρ_k	half downrange reach of the parallelogram footprint of delivery vehicle k
ρ_{km}	half downrange reach of the m^{th} potential parallelogram footprint for delivery vehicle k
σ	ratio of downrange reach to crossrange reach of true footprint
$\phi_{\ell k}$	value accrued at $\underline{\xi}_{\ell k}$
$\overline{\phi}_{\ell k}$	"added value" for including $\gamma_{\ell k}$
$\underline{\omega}_o$	dual variables corresponding to convexity of S_k
$\underline{\omega}_1$	dual variables corresponding to single coverage constraints

EPILOGUE

The goal of including the dispersion constraints while simultaneously finding optimal (or near optimal) assignments of weapons to DGZs is, indeed, valuable. This, however, introduces a problem of great computational size. While we have produced here a model and solution methodology, it is yet to be determined whether any computer system possesses the size and speed to generate significant results. Thus, the next step appears to be the coding of an algorithm for execution on an increasingly large sequence of test problems. In such a way we may determine the magnitude and scope of problems which can be approached with presently available computer technology. It appears evident that every computational trick and device must be explored in order to place the larger problems on a computer and have them executed in reasonable time; efficient coding and virtual storage are identified as key elements of this task.

Even if the problem with the entire set of weapons and DGZs allocated to a given exercise is too large for significant computational results, it is likely that a decomposition to smaller subproblems with a smart recombination will produce good suboptimal results. Also, if at the first stage we identify a good assignment without dispersion, then within a target-rich environment it may not be difficult to perturb the assignment to obtain a good assignment which also satisfies the dispersion constraints. That is, problem (21-24) is amenable to linear programming algorithms; if we apply branch-and-bound using constraints (29), because of the special structure of the polytope of feasible solutions to (21-24) we may quickly find an optimal solution of (21-24, 29); then only the perturbation to satisfy dispersion remains.

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When one is unable to place the available nuclear weapons to cover every installation so as to obtain a minimum specified damage on each installation, the supply of weapons may be considered as a scarce resource which must be assigned to designated ground zeros (DGZ) in such a way as to optimize a prescribed objective. A mixture of weapon and delivery vehicle types (including MIRVs) is considered.

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20. Abstract continued.

For any fixed layout of DGZs in a tiered aimpoint system, any deployment of delivery vehicles, and any value system, an analytical model using mixed integer/linear programming is proposed to consider the target-rich environment case. The problem of dispersion of MIRV weapons is resolved by approximating the true footprint by a parallelogram footprint. Identification of decompositions of the problem by delivery vehicles enabled the construction of dual and primal solution algorithms which converge to an optimal assignment. The solution algorithm may be used to give broad consideration to many hypothetical alternatives, to provide a starting point from which a detailed blueprint for actual assignments by strategic units is constructed, or to generally evaluate present assignment schemes.